MID-TERM EXAM | ISI BANGALORE | FEBRUARY 2017

B MATH ALGEBRA II | 100 POINTS

Notes.
(a) Justify all your steps.
(b) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.
(c) By default, F denotes a field.

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1. [16 points] Find a basis of the solution space for the system $A\vec{x} = \vec{0}$ where A and $\vec{x}$ are given by

$$A = \begin{pmatrix} 0 & 1 & 3 & 0 & 5 & 0 & 7 & 9 \\ 0 & 0 & 0 & 1 & 8 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 3 \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix}$$

You must briefly justify (either by quoting a suitable result or by explicit calculation) why the vectors you produce form a basis of the solution space.

2. [16 points] Let $v_1, \dots, v_n$ be vectors in $\mathbb{Q}^m$ having all integer entries. Let p be a prime number and let $\overline{v_1}, \dots, \overline{v_n}$ denote the vectors in $\mathbb{F}_p^m$ obtained by reducing the entries of v_i modulo p. If $\overline{v_1}, \dots, \overline{v_n}$ span $\mathbb{F}_p^m$ prove that $v_1, \dots, v_n$ span $\mathbb{Q}^m$.

3. [20 points] Let V, V' be vector spaces over a field F and let $W \subset V$ and $W' \subset V'$ be subspaces. Prove that there exists an onto linear map $T \colon V \to V'$ with $T(W) = W'$ if and only if the following two inequalities hold:

$$\dim(W) \geq \dim(W'), \qquad \operatorname{codim}_V(W) \geq \operatorname{codim}_{V'}(W').$$

4. [16 points] Let $W_1, \dots, W_k$ be subspaces of a vector space V.
   (i) Define what it means for $W_1, \dots, W_k$ to be independent subspaces in V.
   (ii) If

   $$W_1 \cap W_2 = (0), \quad (W_1 + W_2) \cap W_3 = (0), \quad \dots, \quad (W_1 + W_2 + \cdots + W_{k-1}) \cap W_k = (0),$$

   then prove that $W_1, \dots, W_k$ are independent.

5. [16 points]
   (i) Let X, Y be matrices of size $m \times r$ and $r \times n$ respectively. Prove that the rank of XY is at most r.
   (ii) Let A be an $m \times n$ matrix of rank r. Prove that there exist matrices X, Y of size $m \times r$ and $r \times n$ respectively such that $A = XY$.

6. [16 points] Let V denote the vector space of all polynomials of degree at most 3 in a variable x over $\mathbb{R}$. Let $\Delta\colon V \to V$ be given by $\Delta(f(x)) := f(x+1) - f(x)$.

(i) Verify that Δ is a linear operator.

(ii) Write down the matrix of Δ for the ordered basis of V given by $1, x, x^2, x^3$.

(iii) Write down the matrix of Δ for the ordered basis of V given by

$$\binom{x}{0} = 1, \quad \binom{x}{1} = x, \quad \binom{x}{2} = x(x-1)/2, \quad \binom{x}{3} = x(x-1)(x-2)/6.$$